Introduction to the Big Ramsey Degrees

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Prague

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Ramsey Theorem

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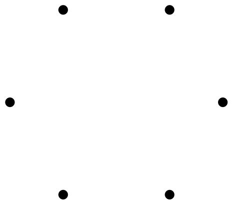

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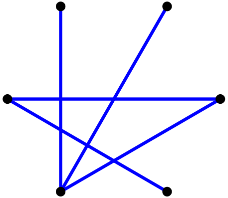

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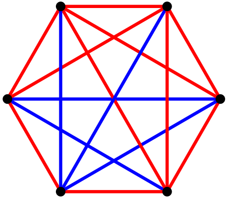

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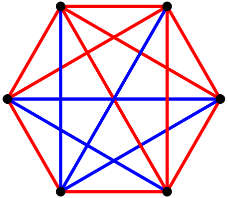

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For every $p, n, k \geq 1$ *there exists* $N > 1$ *such that* $N \longrightarrow (n)^p_k$.

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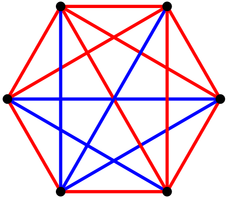

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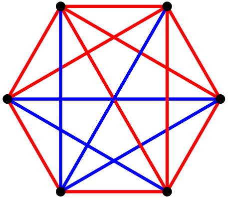
Erdös–Rado partition arrow

$N \longrightarrow (n)^p_k$: For every partition of p-element subsets of X, $|X| \geq N$ into k classes (colours) there exists $Y \subseteq X$, $|Y| = n$ such that all p-element subsets of Y belongs to a single partition. (Y is monochromatic.)

Ramsey Theorem

$p = 2, n = 3, k = 2, N = 6$



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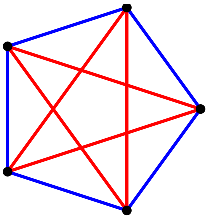

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$N \longrightarrow (n)^p_{k,t}$: For every partition of $\binom{\omega}{p}$ into k classes (colours) there exists $X \in \binom{\omega}{\omega}$ such that $\binom{X}{p}$ belongs to at most t parts.

($t = 1$ means that $\binom{X}{p}$ is monochromatic.)

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In 1970's a concept of structural Ramsey theory was introduced. A Ramsey theorem can be seen as a theorem about the class of linear orders.

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of structure **A** to structure **B**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

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Sierpiński: not true for $|O| = 2$.

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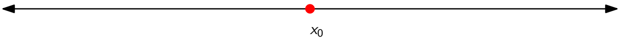

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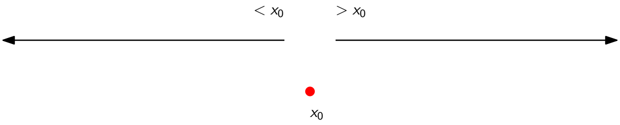

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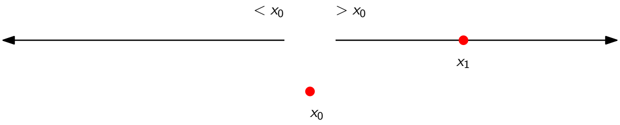

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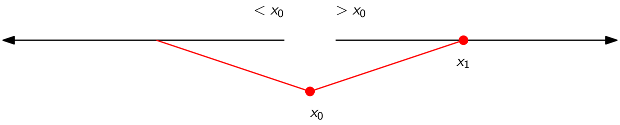

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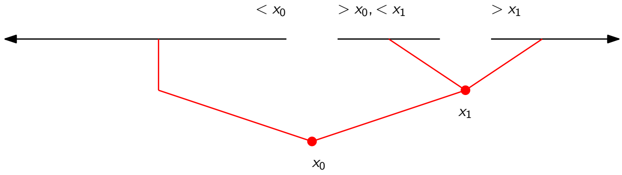

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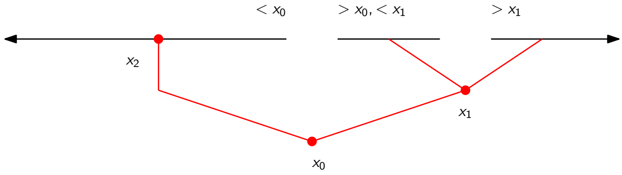

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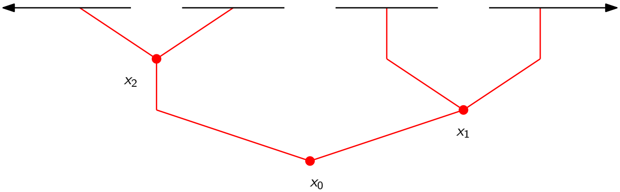

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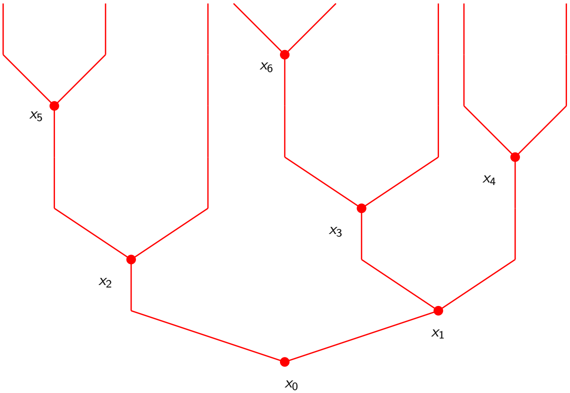

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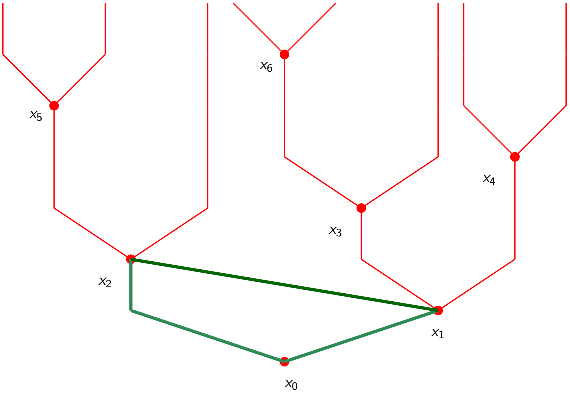

Colour of k-tuple = shape of meet closure in the tree

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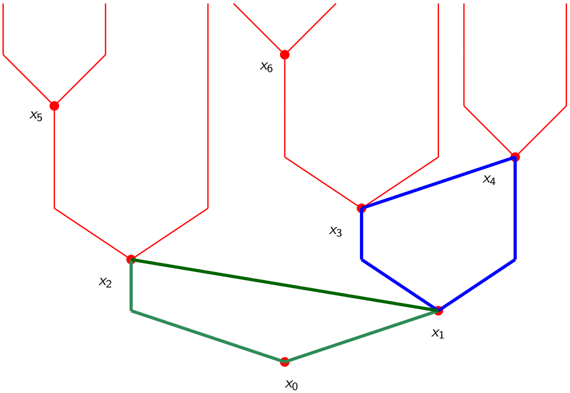

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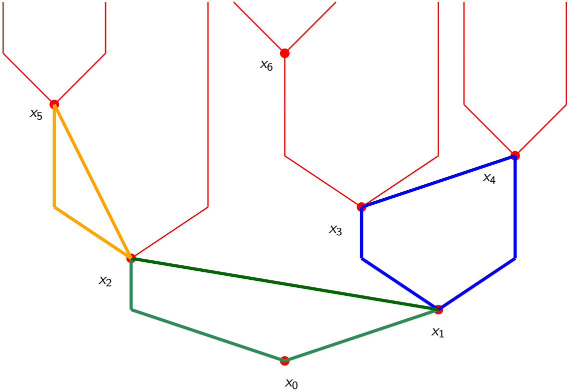

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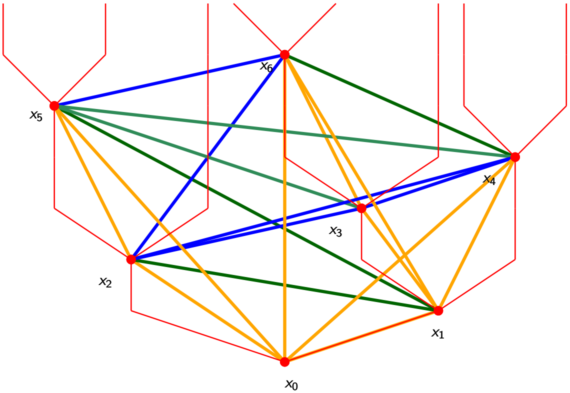

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Big Ramsey Degrees of $(\mathbb{Q}, \leq)$

In late 1960's Laver developed method of finding copies of $\mathbb{Q}$ in $\mathbb{Q}$ with bounded number of colours using Milliken's tree theorem.

Theorem (Devlin, 1979)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q},\leq) \longrightarrow (\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

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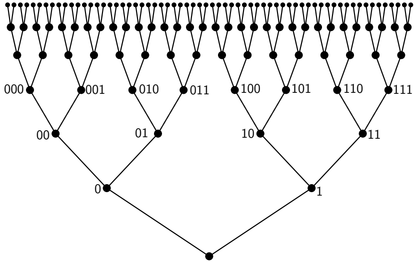
$$T(1) = 1,\ T(2) = 2,\ T(3) = 16,\ T(4) = 272,$$

$$T(5) = 7936,\ T(6) = 353792,\ T(7) = 22368256$$

Trees (terminology)

- A tree is a (possibly empty) partially ordered set $(T, <_T)$ such that, for every $t \in T$, the set $\{ s \in T : s <_T t \}$ is finite and linearly ordered by $<_T$. All trees considered are finite or countable.
- All nonempty trees we consider are rooted, that is, they have a unique minimal element called the root of the tree.
- An element $t \in T$ of a tree T is called a node of T and its level, denoted by $|t|_T$, is the size of the set $\{ s \in T : s <_T t \}$.
- We use $T(n)$ to denote the set of all nodes of T at level n,
- For $s, t \in T$, the meet $s \wedge_T t$ of s and t is the largest $s' \in T$ such that $s' \leq_T s$ and $s' \leq_T t$.
- The height of T, denoted by $h(T)$, is the minimal natural number h such that $T(h) = \emptyset$. If there is no such number h, then we say that the height of T is ω.



$2^{<\omega}$

Subtrees and strong subtrees

- A subtree of a tree T is a subset T' of T viewed as a tree equipped with the induced partial ordering such that $s \wedge_{T'} t = s \wedge_T t$ for each $s, t \in T'$.
- Given a tree T and nodes $s, t \in T$ we say that s is a successor of t in T if $t \leq_T s$.
- The node s is an immediate successor of t in T if $t <_T s$ and there is no $s' \in T$ such that $t <_T s' <_T s$.
- We denote the set of all successors of t in T by $\mathrm{Succ}_T(t)$ and the set of immediate successors of t in T by $\mathrm{ImmSucc}_T(t)$.

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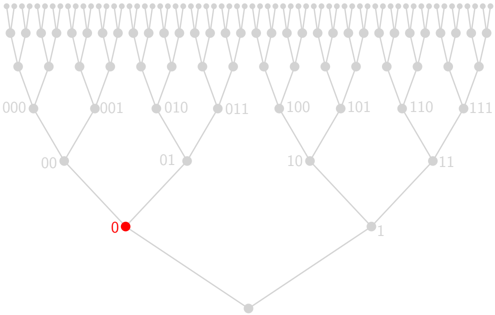
Definition (Strong subtree)

A subtree S of a tree T is a strong subtree of T if either S is empty, or S is nonempty and satisfies the following three conditions.

1. The tree S is rooted and balanced.
2. Every level of S is a subset of some level of T, that is, for every $n < h(S)$ there exists $m \in \omega$ such that $S(n) \subseteq T(m)$.
3. For every non-maximal node $s \in S$ and every $t \in \mathrm{ImmSucc}_T(s)$ the set $\mathrm{ImmSucc}_S(s) \cap \mathrm{Succ}_T(t)$ is a singleton.

Strong subtree



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A subtree $\mathbf{S} \subseteq \mathbf{T}$ is a subset closed for meets. It is strong if:

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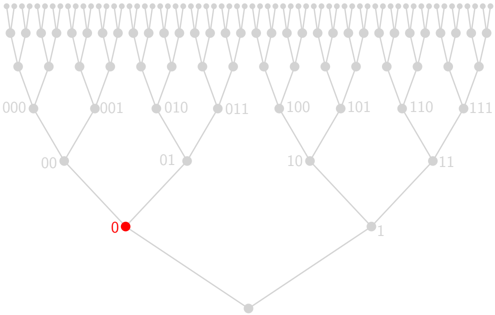

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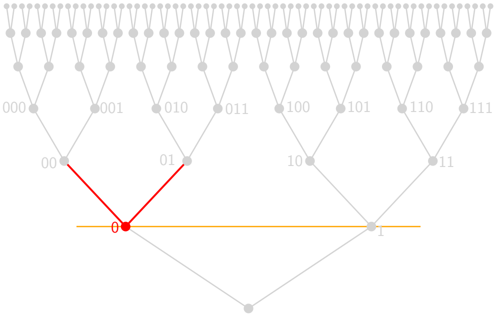

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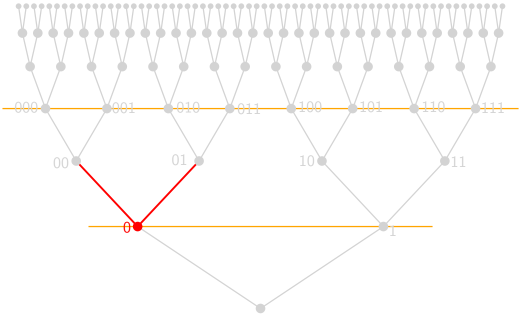

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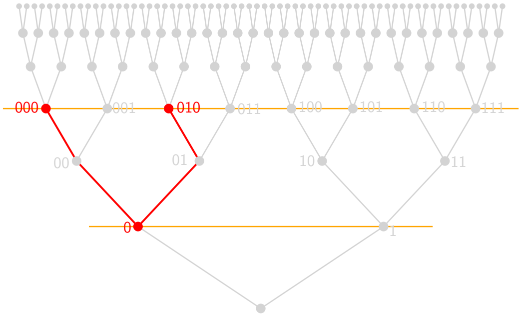

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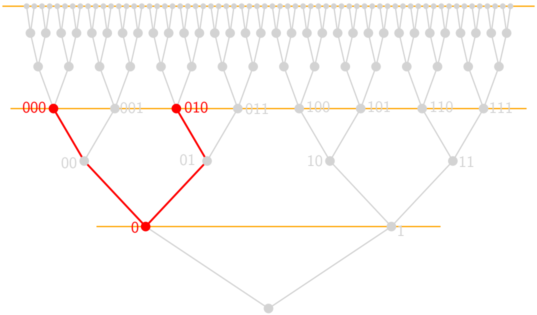

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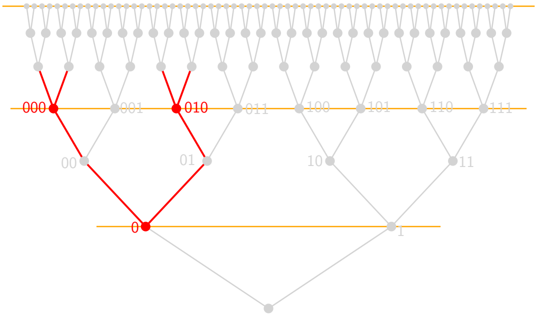

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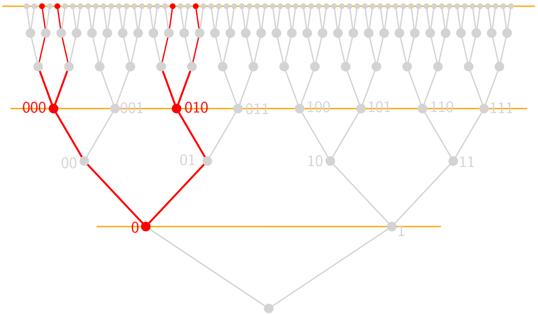

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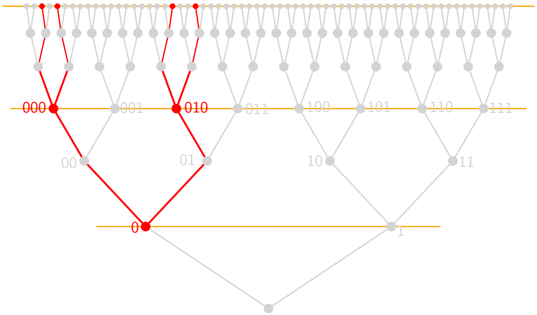

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3. Every level of S is a subset of some level of T.
4. S either has no leaves or all are at the same level.

Ramsey-type theorem for strong subtrees

Let T be a tree and $k \in \omega + 1$. We use $\mathrm{Str}_k(T)$ to denote the set of all strong subtrees of T of height k.

Theorem (Milliken 1979)

For every rooted, balanced and finitely branching tree T of infinite height, every $k \in \omega$ and every finite colouring of $\mathrm{Str}_k(T)$ there is $S \in \mathrm{Str}_\omega(T)$ such that the set $\mathrm{Str}_k(S)$ is monochromatic.

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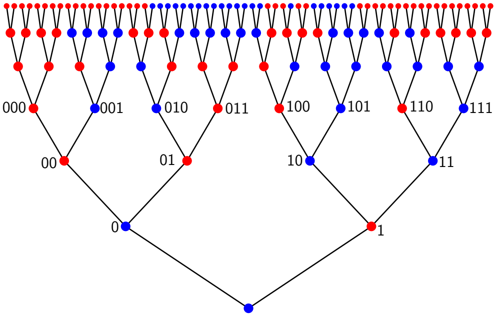
The difficult case to prove is $k = 1$ (Halpern–Läuchli Theorem, 1966)



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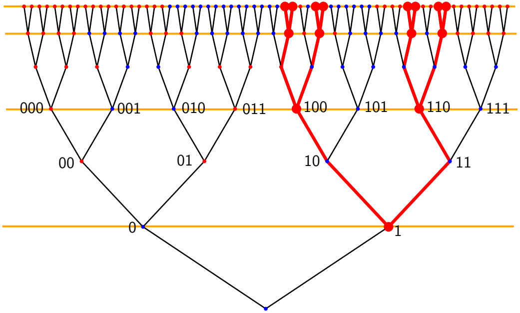

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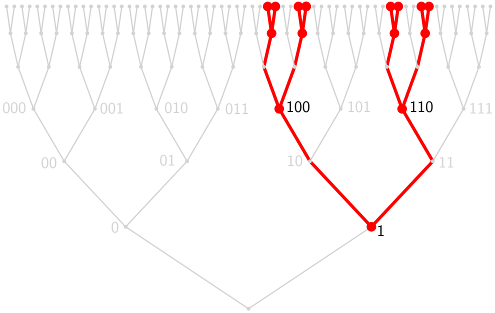

Notice that for regularly branching tree the strong subtree is isomorphic to the original tree.

Big Ramsey degrees using Milliken tree theorem

We aim to prove:

Theorem (Laver, late 1969)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q},\leq)\longrightarrow(\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

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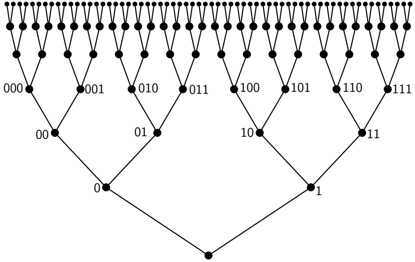

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1. Nodes of the binary tree $2^{<\omega}$ ordered "from left to right" yields $(\mathbb{Q},\leq)$.
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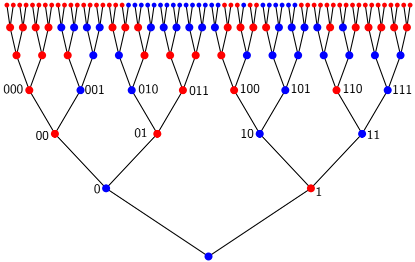

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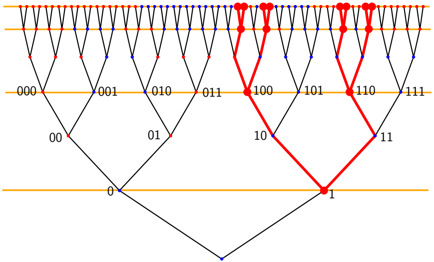
4. Strong subtree is isomorphic to the original binary tree and also isomorphic to $(\mathbb{Q}, \leq)$ when ordered lexicographically
   $\implies$ we found the monochromatic copy!



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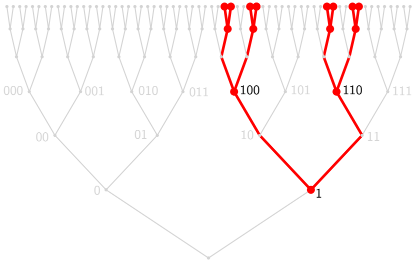

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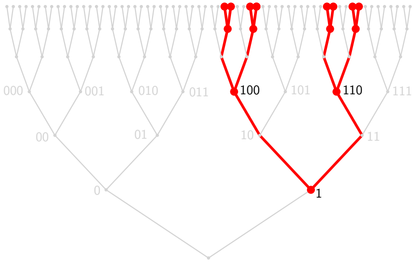
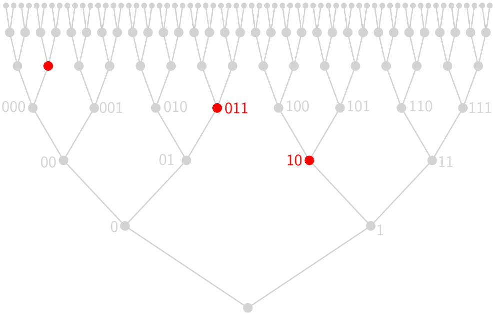


If $|O| = n > 1$ we transfer colourings of n-tuples of nodes to colouring of strong subtrees.

Envelopes of subsets



- Given a subset X of tree T, an envelope is any strong subtree S of T containing X.
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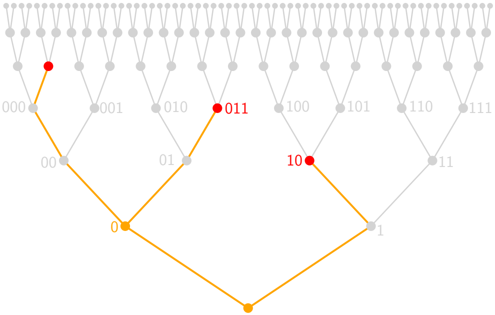

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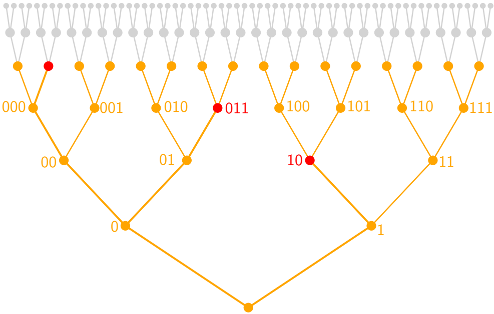

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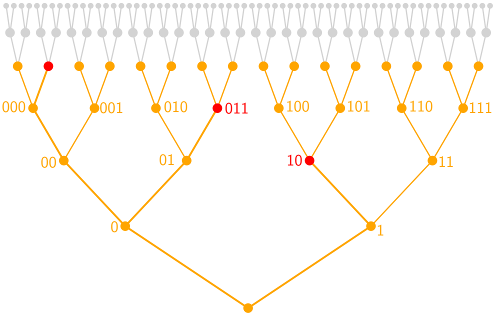

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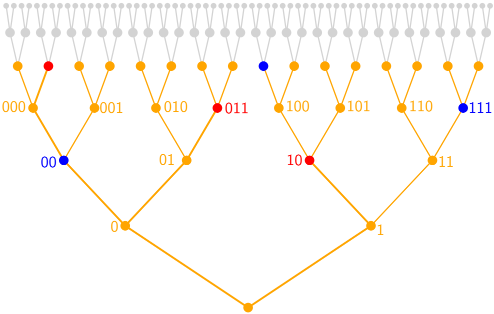

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A minimal envelope of every finite X has height at most $2|X| - 1$.

Multiple choices of X may lead to a same envelope. We speak of different embedding types within a given envelope.

Big Ramsey degrees using Milliken tree theorem

Now we can finish proof of:

Theorem (Laver, late 1969)

$$\forall_{(O,\leq_O)\in\mathcal{O}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (\mathbb{Q},\leq) \longrightarrow (\mathbb{Q},\leq)^{(O,\leq_O)}_{k,T}.$$

Proof.

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2. $T(n)$ is the number of embedding types of n-tuples in the binary tree.
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4. For each embedding type construct colouring of envelopes and pass to a monochromatic subtree by the application of Milliken tree theorem.
5. The resulting copy will have at most $T(n)$ different colours

□

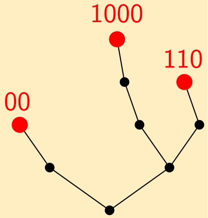
Big Ramsey degrees of $(\mathbb{Q}, \leq)$ are finite!

Devlin types

Definition (Devlin types)

$A \subseteq 2^{<\omega}$ a Devlin embedding type iff it is an antichain and for every $0 \leq \ell < \max_{a \in A} |a|$ precisely one of the following happens:

1. Leaf: There exists precisely one $a \in A$ with $|a| = \ell$.
   Moreover for every $b \in A$, $|b| > \ell$ it holds that $b(\ell) = 0$.
2. Branching: There exists $a, b \in A$, $|a|, |b| > \ell$ such that $a(\ell) = 0$, $b(\ell) = 1$ and moreover for every $c \in A$, $|c| \geq \ell$ whose iniital segment of length ℓ is different form b it holds that $|c| > \ell$ and $c(\ell) = 0$.



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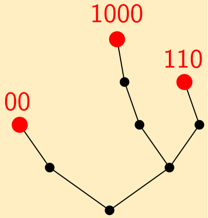

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$$t_n = \tan^{(2n-1)}(0) = \sum_{\ell=1}^{n-1} \binom{2n-2}{2\ell-1} t_\ell \cdot t_{n-1} \text{ with } n_1 = 1$$

This is a well known sequence (of the odd tangent numbers).

Victory!

We characterised the big Ramsey degrees of rationals and gave a closed-form formula.

Big Ramsey degrees of **R**

Definition

A (countable) structure **A** is (ultra) homogeneous if every its partial isomorphism extends to an automorphism.

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$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{T=T'(\mathbf{A})\in\omega}\forall_{k\geq 1} : \mathbf{R} \longrightarrow (\mathbf{R})^{\mathbf{A}}_{k,T}.$$

This theorem was published by Sauer in 2006 and also appears in Todorčević' Introduction to Ramsey spaces. Values of $T'(\mathbf{G})$ were characterised by Laflamme–Sauer–Vuksanović in 2010.

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A finitary version is (probably more) famous!

Theorem (Nešetřil–Rödl 1977, Abramson–Harington 1978)

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{t=t(\mathbf{A})\in\omega}\forall_{\mathbf{B}\in\mathcal{G},k\geq 1}\exists_{\mathbf{C}} \in \mathcal{G} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}.$$

Understanding the unavoidable colourings

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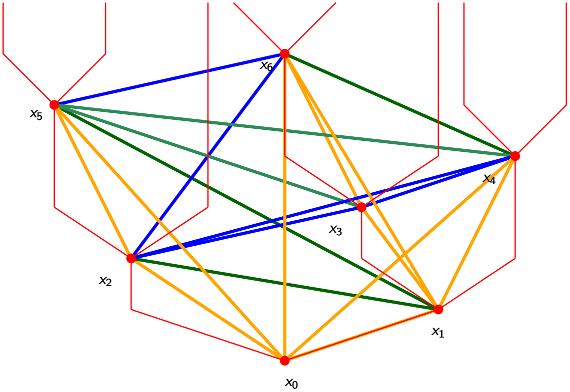
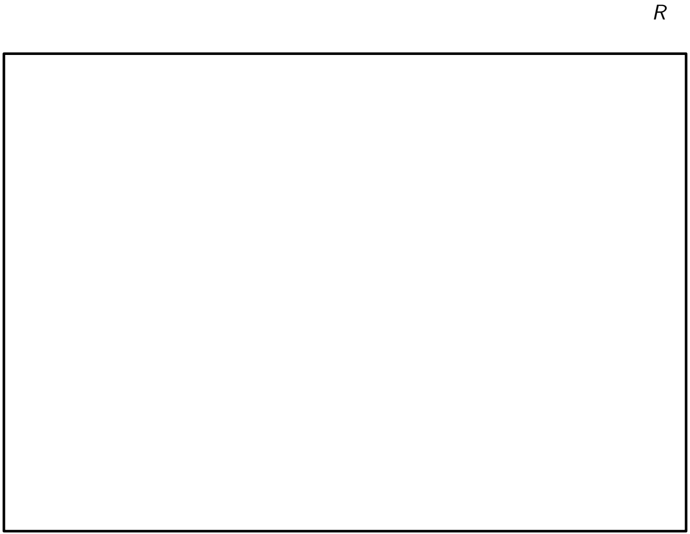


For $(\mathbb{Q}, \leq)$ we have the Sierpiński colourings.
Can we do something similar for the Rado graph?

Understanding the unavoidable colourings of Rado graphs



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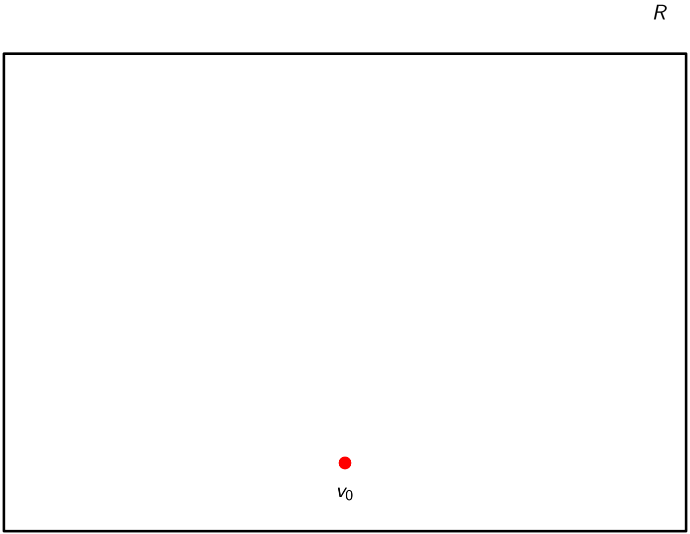

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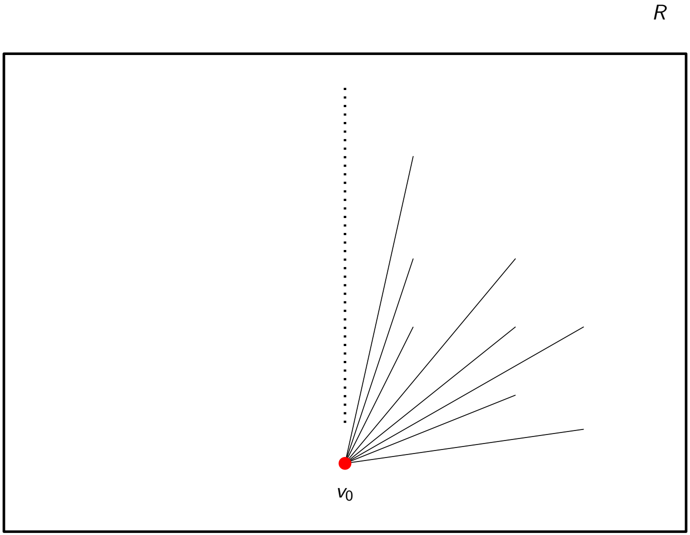

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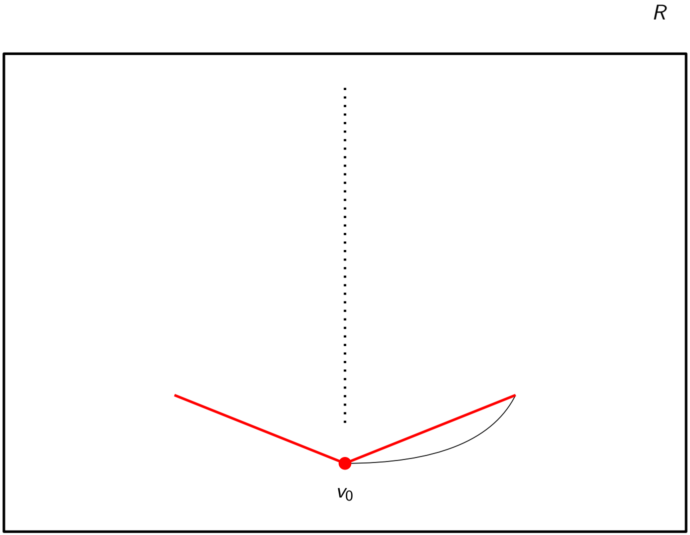

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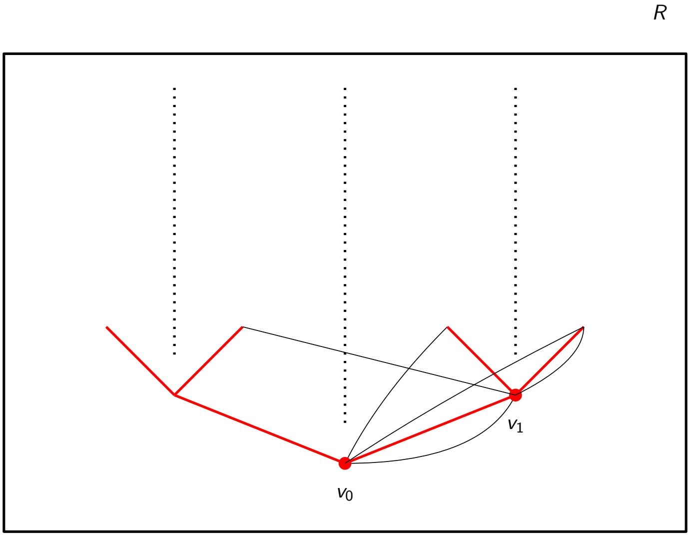

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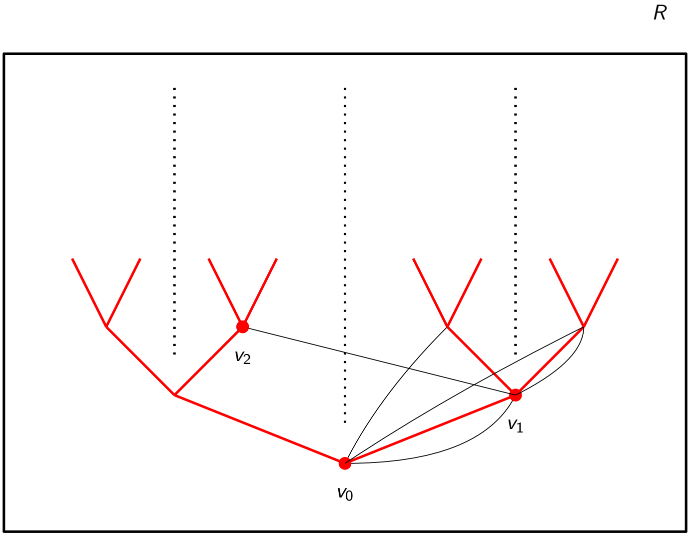

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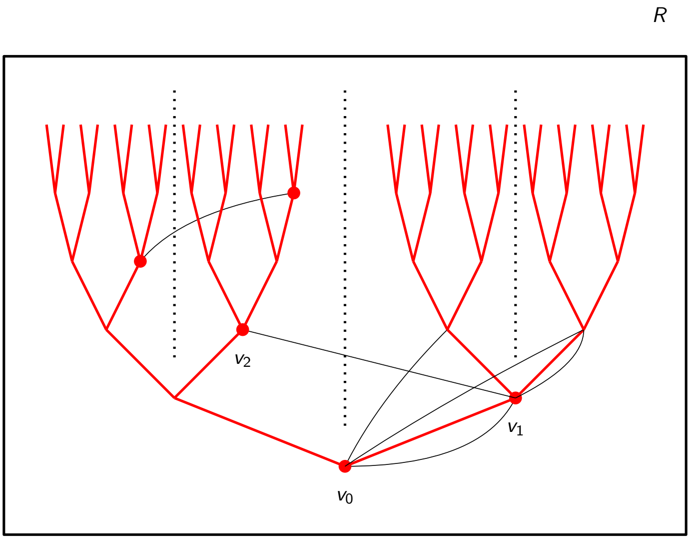

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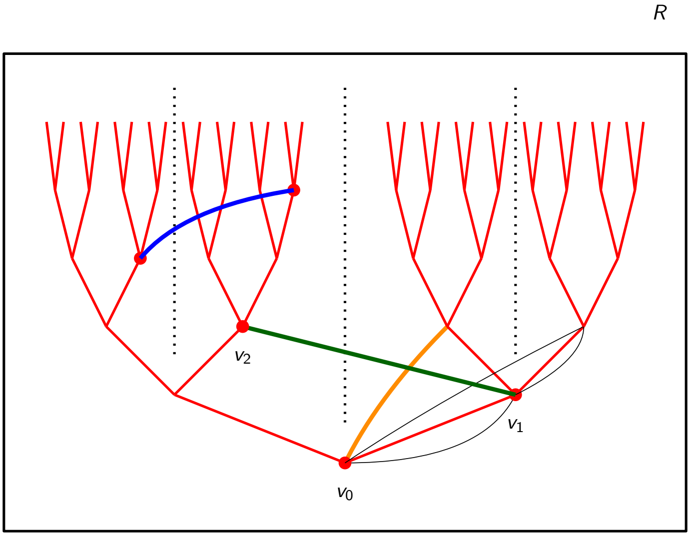
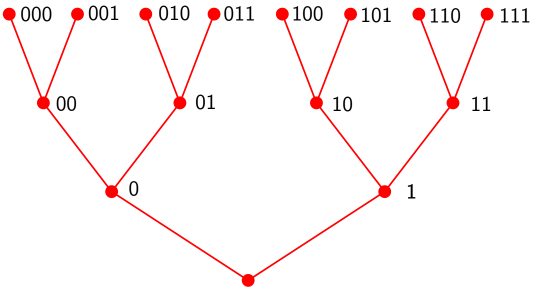


Passing number graph



Definition (Graph G)

We will consider graph **G**:

1. Vertices: $2^{<\omega}$
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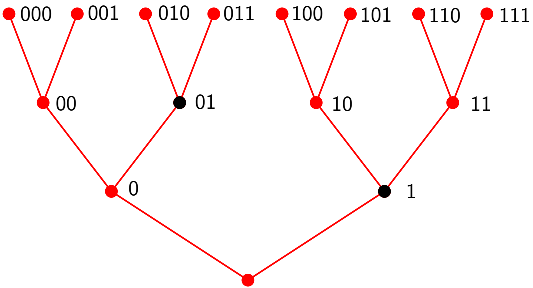

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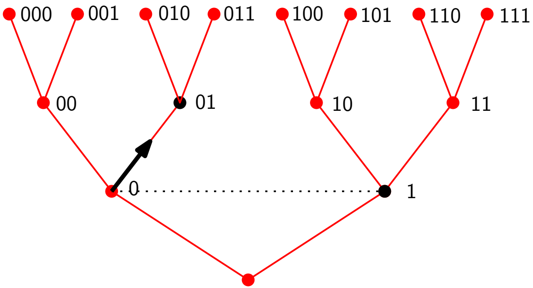

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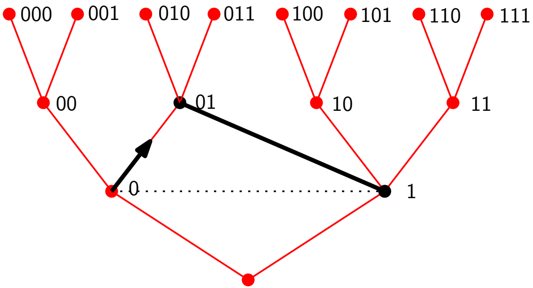

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The upper bound

Lemma

G *is universal: the Rado graph* **R** *embeds to* **G**.

Proof.

Assume that the vertex set of **R** is ω. The vertex $i \in \omega$ then corresponds to a sequence a of length i with $a(j) = 1$ if and only if $i \sim j$. □

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Lower bounds needs a bit more care.

Known big Ramsey results by proof techniques

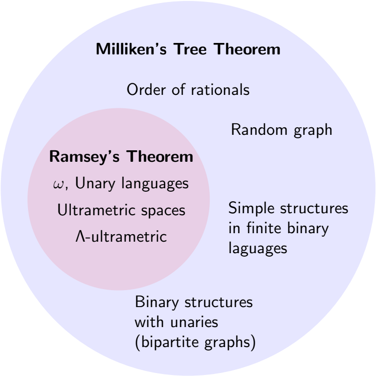
Ramsey's Theorem

ω, Unary languages

Ultrametric spaces

Λ-ultrametric

Known big Ramsey results by proof techniques



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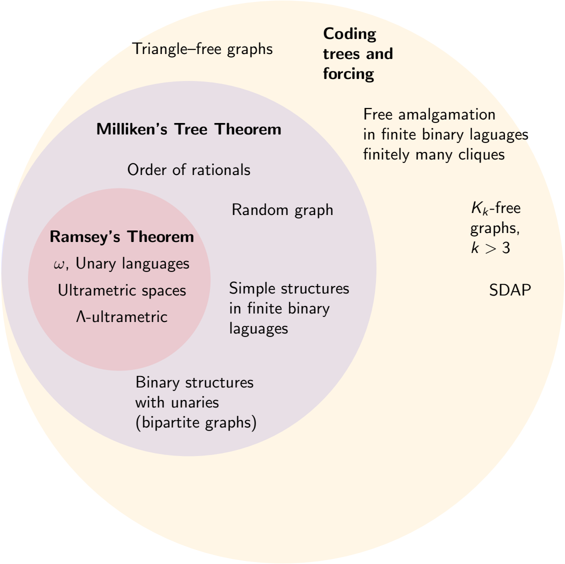

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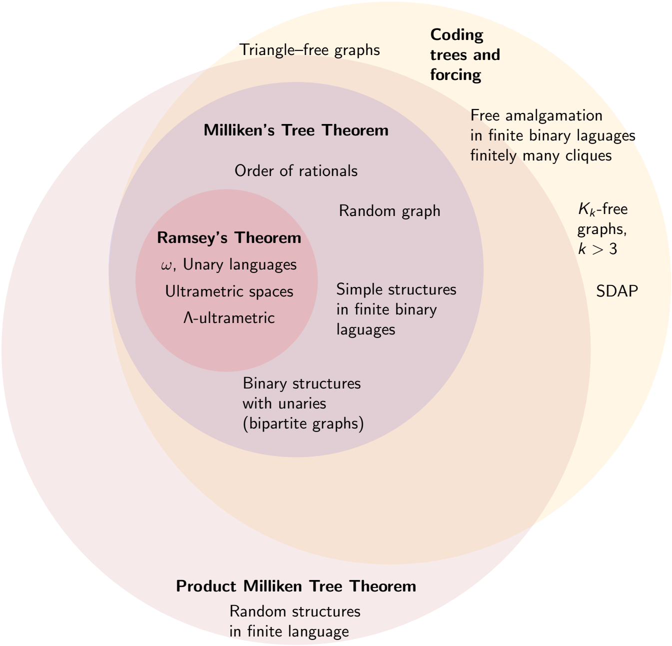

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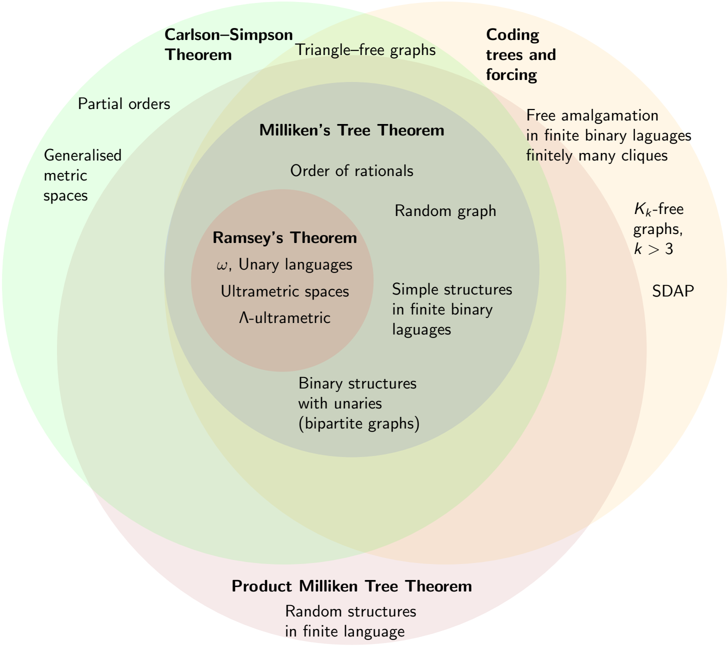

Big Ramsey degrees of the universal triange-free graph

Let $\mathcal{T}$ be the class of all finite triangle-free graphs.

We aim to give an easy proof of:

Theorem (Dobrinen 2020)

Every (countable) universal triangle-free graph $\mathbf{T}$ *has finite big Ramsey degrees:*

$$\forall_{\mathbf{A}\in\mathcal{T}}\exists_{T=T(\mathbf{A})\in\omega}\forall_{k\geq 1} : \mathbf{A} \longrightarrow (\mathbf{T})^{(\mathbf{A})}_{k,T}.$$

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3. Zucker streamlined and generalized the proof to all classes of structures in finite binary languages described by finitely many forbidden irreducible substructures.

A. Zucker. On big Ramsey degrees for binary free amalgamation classes. Advances in Mathematics, 408 (2022), 108585. 25 pages.

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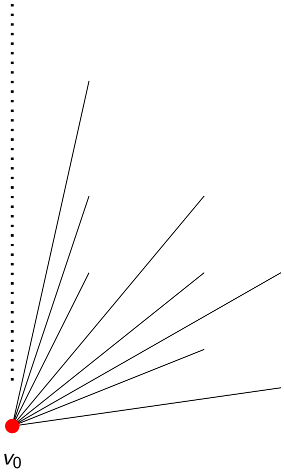

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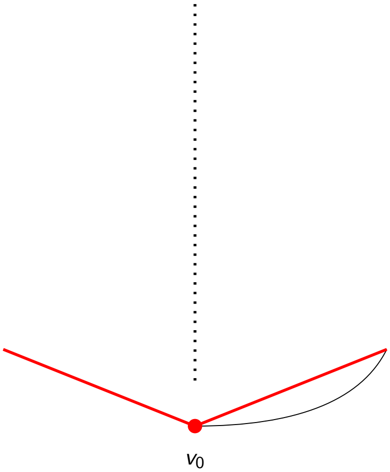

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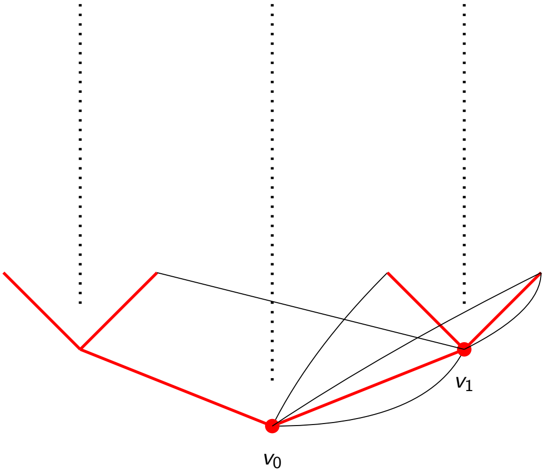

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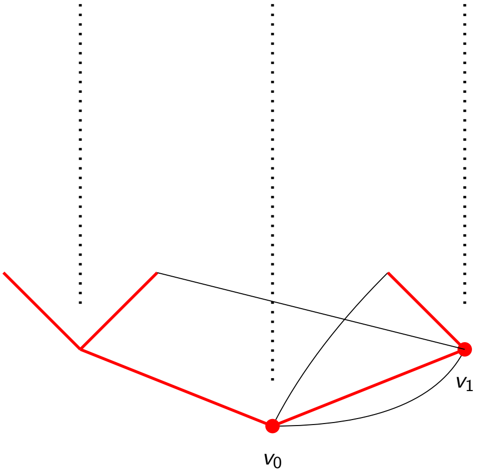

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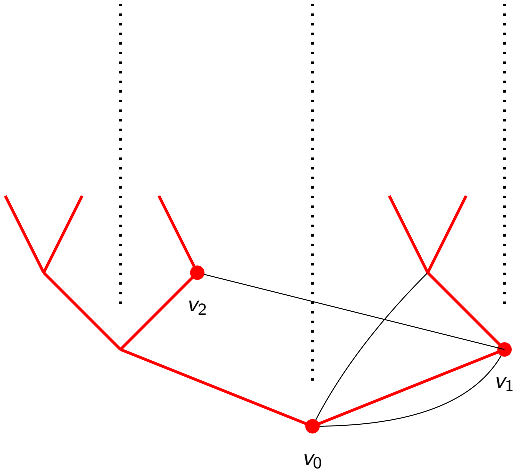

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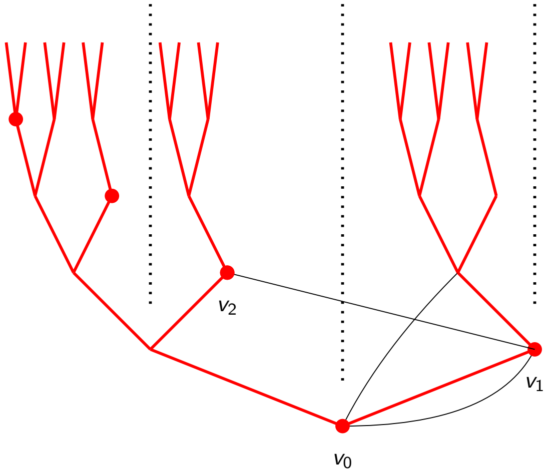

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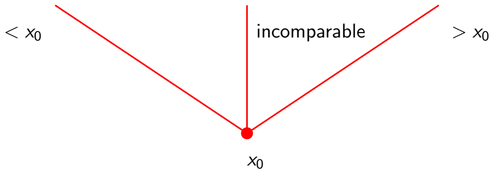

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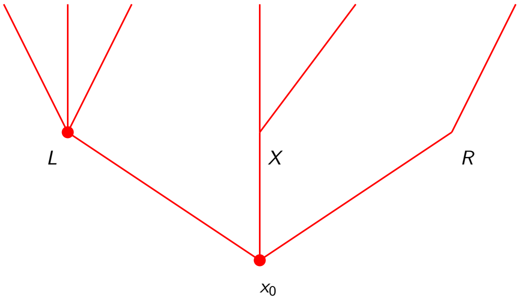

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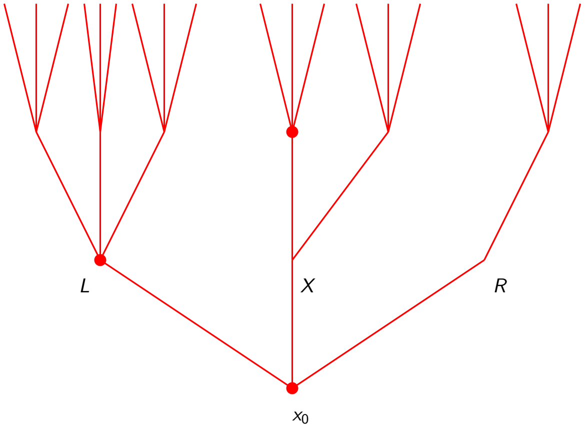

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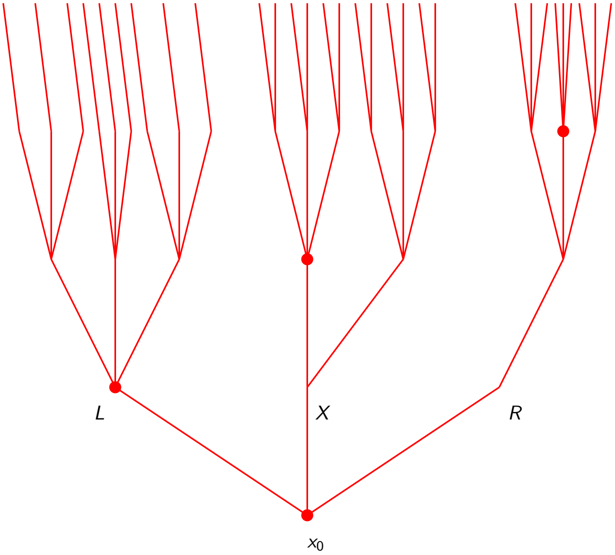

Parameter words

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Given a finite alphabet Σ and $k \in \omega + 1$, a k-parameter word is a (possibly infinite) word W in alphabet $\Sigma \cup \{\lambda_i : 0 \leq i < k\}$ such that $\forall i \in k$ word W contains λ_i and for every $j \in k - 1$, the first occurrence of λ_{j+1} appears after the first occurrence of λ_j.

Example (2-parameter word)

$\Sigma = \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}$.

$$\mathrm{LRL}\lambda_0\lambda_0\mathrm{X}\lambda_1\lambda_0\mathrm{R}$$

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For set S of parameter words and a parameter word W:

$$W(S) = \{W(U) : U \in S\}.$$

Ramsey theorem for parameter words

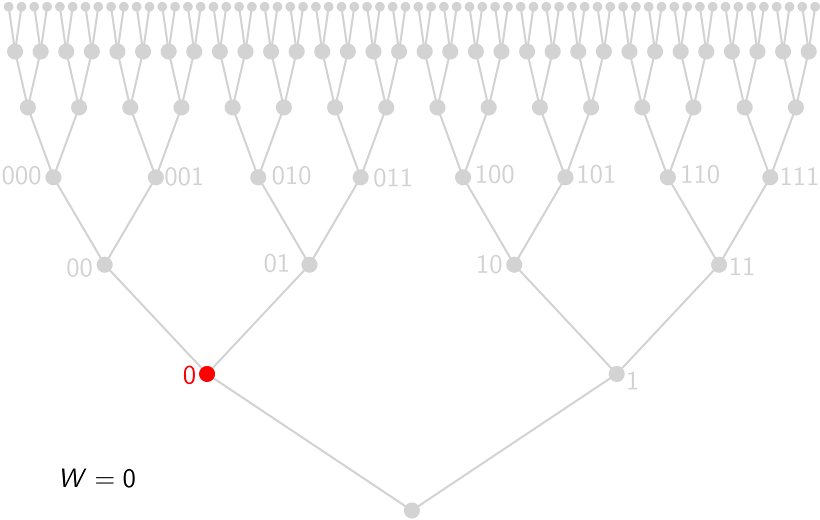
The following infinitary version of Graham–Rothschild Theorem is a direct consequence of the Carlson–Simpson theorem. It was also independently proved by Voight in 1983 (apparently unpublished):

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

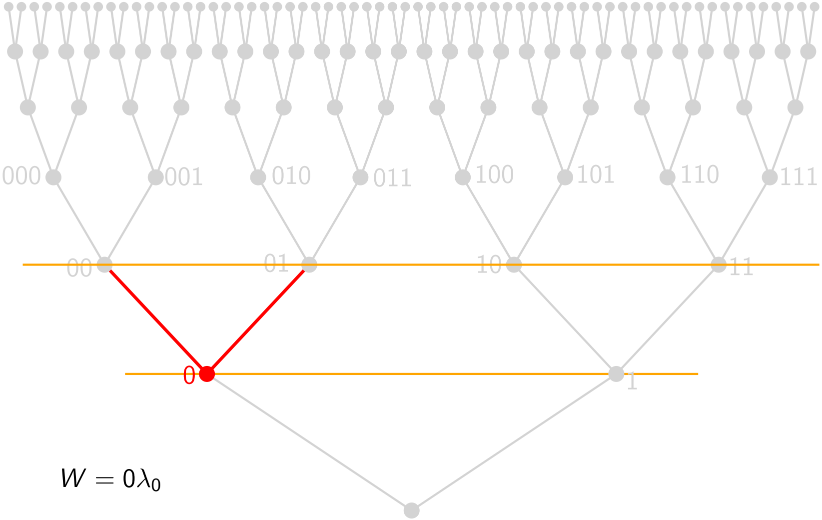
By W being monochromatic we mean that for every pair of k-parameter words U, V the colour of $W(U)$ is the same as colour of $W(V)$.

Parameter words as subtrees



$W = 0$

Parameter words as subtrees



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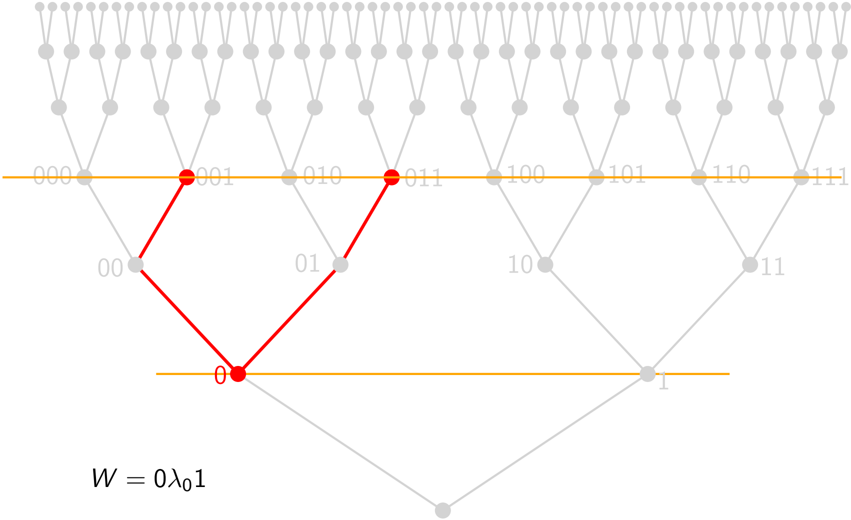

$W = 0\lambda_0 1$

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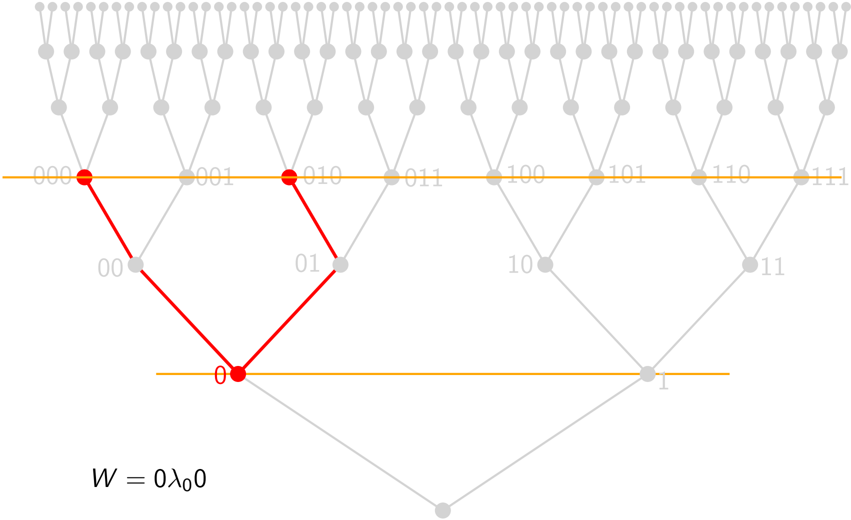

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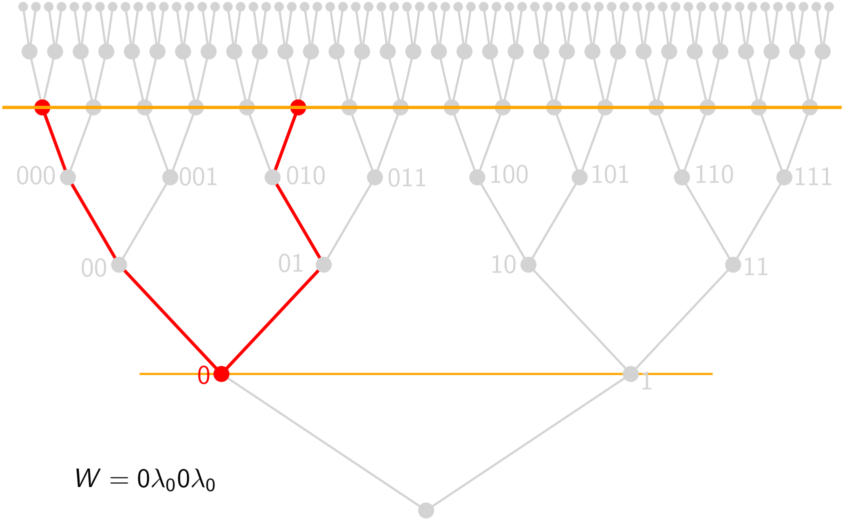

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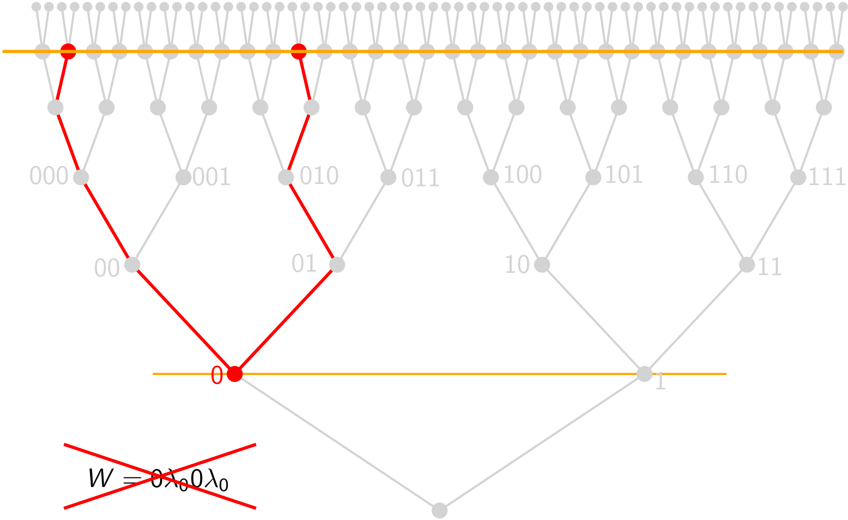

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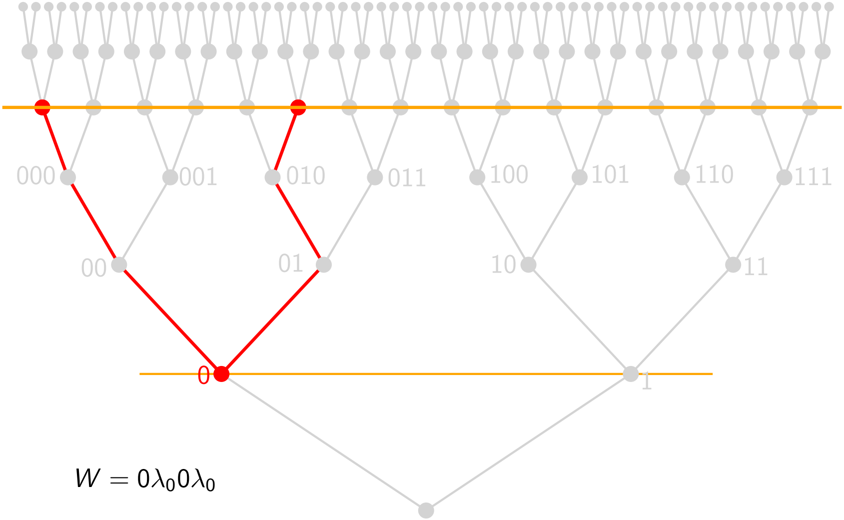

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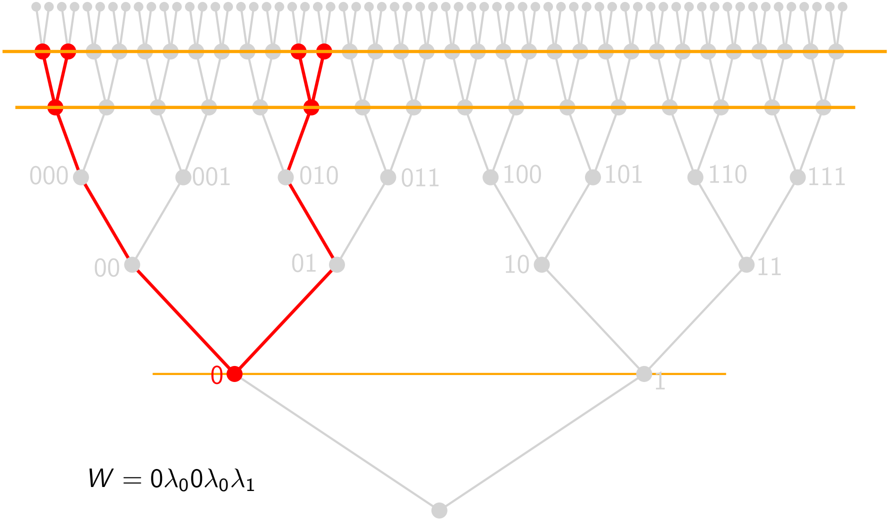

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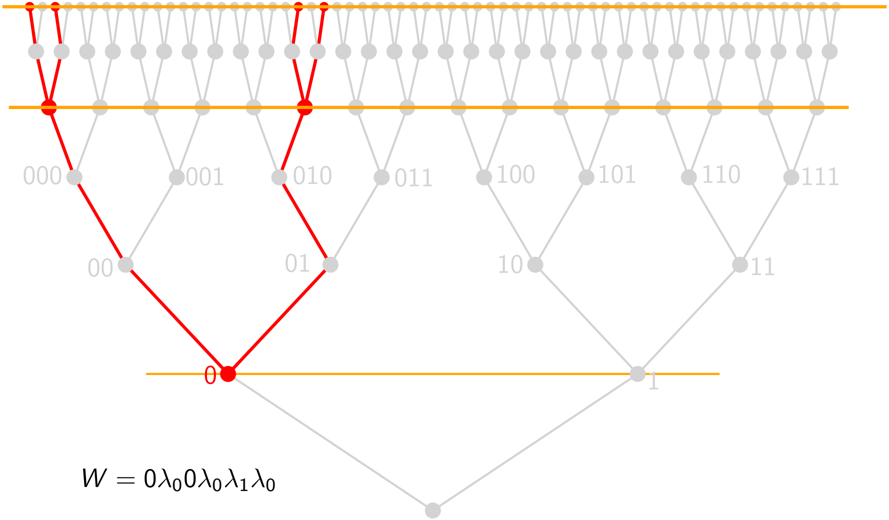

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Definition

Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set k-parameter words, an envelope of S is every n-parameter word W (for some $n \geq k$) such that

$$\forall_{w \in S} \exists_u W(u) = w.$$

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Proof.

U	$=$	0	1	1	0	1	
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Proof.

$$\begin{array}{lcllllll} U & = & 0 & 1 & 1 & 0 & 1 & \\ V & = & 0 & 0 & 1 & 1 & 0 & 1 \\ \text{Envelope:} & & 0 & \lambda_0 & 1 & \lambda_1 & & \end{array}$$

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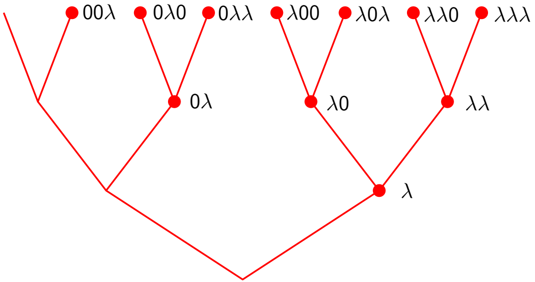
□

Triangle-free graph on 1-parameter words



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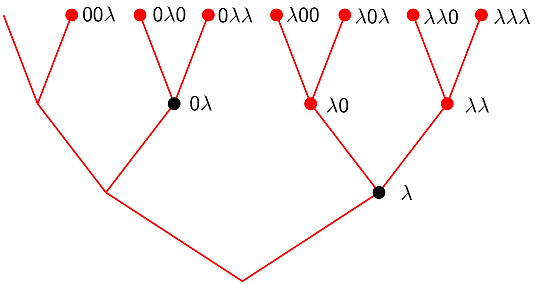

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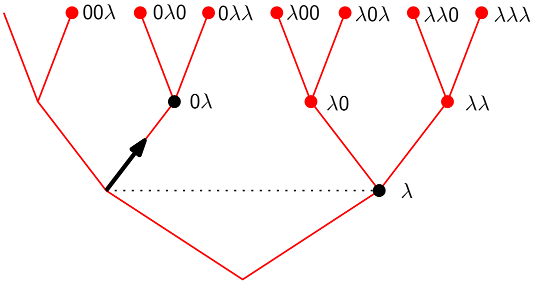

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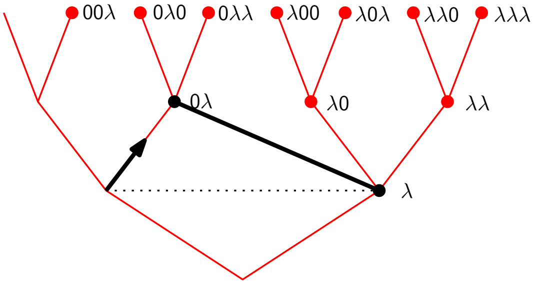

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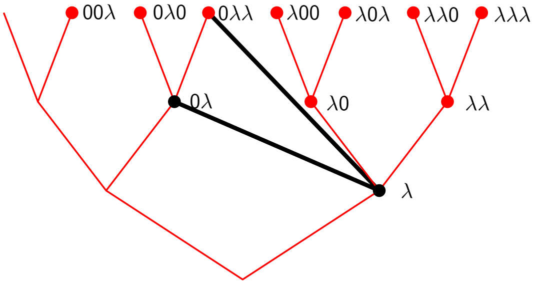

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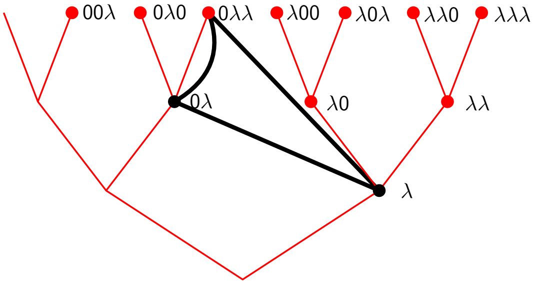

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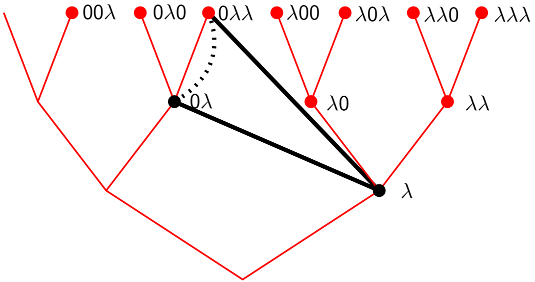

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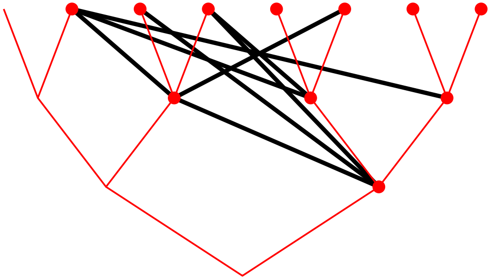
Triangle-free graph on 1-parameter words



Put $\Sigma = \{0\}$.

Definition (Triangle-free graph G)

- Vertices of G are all finite 1-parameter words in alphabet Σ.
- For 1-parameter words u, v satisfying $|u| < |v|$ we put $u \sim v$ (u adjacent to v) iff
  1. $V_{|U|} = \lambda$ and
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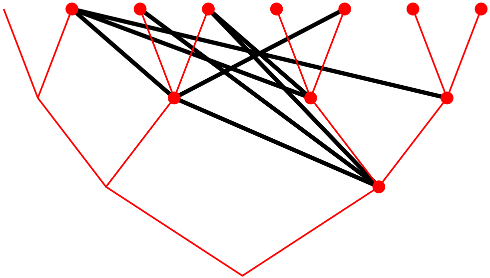

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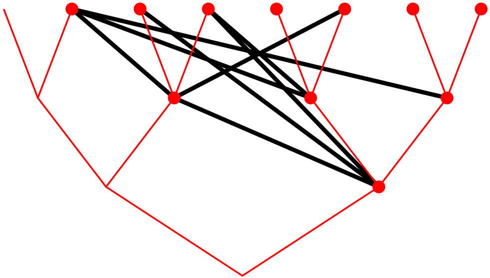

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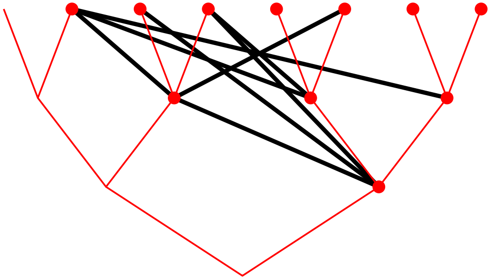
Key observation 1: G is universal triangle-free graph.
Given any triangle-free graph H with vertex set ω assign every $i \in \omega$ word w of length i putting $\forall_{j<i}\ w_j = \lambda$ iff $\{i, j\}$ is an edge of H.

Triangle-free graph on 1-parameter words



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Key observation 1: G is universal triangle-free graph.
Key observation 2: For every pair of 1-parmeter words U and V and every ω-parameter W

$$U \sim V \iff W(U) \sim W(V).$$

Observation

G is a universal triangle-free graph.

Observation

For every infinite-parameter word W it holds that $u \sim v \iff W(u) \sim W(v)$.
(Substitution is also graph embedding on $G \to G$.)

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

Proposition (Envelopes are bounded)

There exists $T(|\Sigma|, s, k)$ such that for every set S of size s of k-parameter words in alphabet Σ there exists an envelope of S with at most $T(|\Sigma|, s, k)$ parameters.

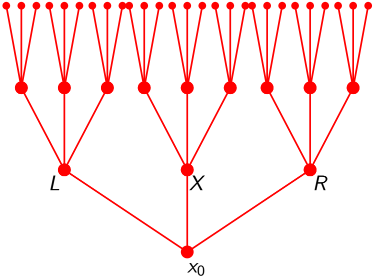
Theorem (Dobrinen 2020)

The big Ramsey degrees of universal triangle-free graph are finite.

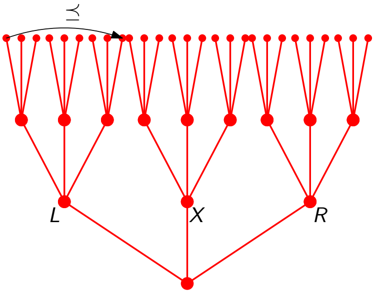
Proof.

Fix graph A and a finite coloring of $\binom{G}{A}$. Because envelopes of copies of A are bounded, apply the theorem above for every embedding type and obtain a copy of G with bounded number of colors. □

Partial order on infinite ternary tree



Partial order on infinite ternary tree



Put $\Sigma = \{\text{L}, \text{X}, \text{R}\}$ and order $\text{L} <_{\text{lex}} \text{X} <_{\text{lex}} \text{R}$.

Definition (Partial order $(\Sigma^*, \preceq)$)

For $w, w' \in \Sigma^*$ we put $w \prec w'$ if and only if there exists $0 \leq i < \min(|w|, |w'|)$ such that

1. $(w_i, w'_i) = (\text{L}, \text{R})$ and
2. for every $0 \leq j < i$ it holds that $w_j \leq_{\text{lex}} w'_j$.

Key observations: $\preceq$ is universal partial order and is stable for substitution.

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Key observations: $\preceq$ is universal partial order and is stable for substitution.

Rest of the proof follows the same way as for triangle-free graph.


SJEZD ČESKÝCH HOSTINSKÝCH V TÁBOŘE 9. VIII 910.

Thank you for the attention

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